Capacity of the Generalized PPM Channel

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Abstract — **We show the capacity of a generalized pulse position modulation (PPM) channel, where the input vectors may be any set that allows a transitive group of coordinate permutations, is achieved by a uniform input distribution. We derive a simple expression for capacity in terms of the Kullback-Leibler distance for the binary case, and find the asymptote in the PPM order. We prove a subadditivity result for the PPM channel and use it to show PPM capacity is monotonic in the order.**

I. Introduction

The deep space optical channel may be modeled as memoryless and operates efficiently at large peak to average power ratios, which may be efficiently implemented with PPM. The set of PPM channel symbols consists of all weight-one n-bit vectors. PPM satisfies the property that each symbol may be obtained as a permutation of the coordinates of any other. We consider a generalization of this, where the input vectors may be any set that allows a transitive group of coordinate permutations. We derive an expression for the capacity of this generalized PPM channel in the binary case, and examine the behavior of the capacity of the PPM channel as a function of the PPM order.

II. Capacity of generalized PPM

Let $p(y|x)$ be the conditional density function of a memoryless channel, $S = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s\}$ a set of length n vectors, $p_{\mathbf{X}}(\cdot)$ a distribution on S, and

$$C_S(n) = \max_{p_{\mathbf{X}}} I(\mathbf{X}; \mathbf{Y}),$$

the capacity of the channel with inputs restricted to S. If there exists a group G of coordinate permutations that fix S such that G acts transitively on S, then we call S a transitive set.

Theorem 1 *If S is a transitive set, then C_S is achieved by a uniform distribution on S.*

With binary inputs, C_S reduces to a simple expression. Let N_1 be the Hamming weight of an element of S and $D(\cdot||\cdot)$ be the Kullback-Leibler distance.

Theorem 2 *On a binary input channel with $p(y|1)/p(y|0) < \infty\ \forall y$,*

$$C_S = N_1 D(p(y|1)||p(y|0)) - D(p(\mathbf{y})||p(\mathbf{y}|\mathbf{0})).$$

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III. Capacity of PPM

Let $C(n)$ be the capacity of a memoryless channel with PPM inputs of length n. Let U and A be the collections of unambiguous and ambiguous outputs when $x = 1$ is transmitted,

$$U = \{y | p(y|1) > 0, p(y|0) = 0\}$$
$$A = \{y | p(y|0) > 0\}.$$

In order to treat the ambiguous and unambiguous outputs separately, define a *reduced channel* $p^*(y|x)$ with output $y \in A$ as follows: $p^*(y|0) = p(y|0)$, $p^*(y|1) = p(y|1)/p(A|1)$, where $p(A|1) = \int_A p(y|1)dy$. Let $C^*(n)$ be the capacity of the reduced channel.

Lemma 1 $C(n) = p(U|1)\log n + p(A|1)C^*(n)$

Hence we can decompose the n-ary PPM channel into an unambiguous channel, which contributes $p(U|1)\log n$ to the capacity, and a reduced channel, with transition probabilities $p^*(y|x)$. In the remainder, we assume the channel is reduced.

Theorem 2 allows a straightforward proof of the asymptotic behavior of the memoryless PPM channel:

Theorem 3 $\lim_{n\to\infty} C(n) = D(p(y|1)||p(y|0))$.

IV. Subadditivity, convexity

We prove the following subadditivity result for the memoryless PPM channel

Theorem 4 *If $n \le m$ then $C(kn) - C(n) \ge C(km) - C(m)$.*

A special case of the theorem is that $C(mn) \le C(m) + C(n)$. We are also now able to say something interesting about 2^k-PPM:

Corollary 1 *For $k = 1, 2, \ldots$, the quantity $C(2^k)/k$ is decreasing in k.*

Corollary 2 *For $k \in \mathbb{N}$, the bits-per-slot capacity of 2^k-PPM on a discrete-time memoryless channel is monotonically decreasing in k.*

A close look at Theorem 4 suggests that the following is likely true: The quantity $(C(k+1) - C(k))/(\log(k+1) - \log k)$ is decreasing in k. Equivalently, the function $C(k)$ is convex-$\cap$ when plotted as a function of $\log k$. As of this writing, we have not proven this, and it remains a conjecture.

References

[1] A. D. Wyner, "Capacity and error exponent for the direct detection photon channel–Part I," *IEEE Transactions on Information Theory*, vol. 34, pp. 1449–1461, Nov. 1988.